Boltzmann Equation Applied to a Problem of Two-Phase Flow

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(Received 17 June 1964)

A limiting case of the interaction between particles in a viscous fluid is described by a linear form of Boltzmann's equation. The distribution function, $f(\mathbf{v})$, is found for small particles falling under the action of gravity and viscous drag, and colliding with large particles. Several approximate procedures for computing $f(\mathbf{v})$ and the connection with flows of gases containing solid particles are examined.

1. INTRODUCTION

IN spite of the long continued interest in various problems involving the behavior of fluids containing solids, relatively little has been accomplished with respect to the influence of direct encounters between particles. The importance of such events is of course contingent upon several circumstances, most obviously, the kinds and concentrations of particles, the motions of the fluid, and the consequent forces acting upon the individual particles. For example, if a fluid containing a very low concentration of particles undergoes a simple slow motion, then not only are the effects of collisions ignorable, but influences of the separate particles are additive; this is the situation first treated by Einstein[1] and since extended by other writers to various kinds of particles and flows. (See the useful survey works by Burgers[2] and by Frisch and Simha.[3]) As the concentration is increased, the disturbances caused by the particles will overlap and one must account for a kind of long-range mutual interaction.

It is clear that for sufficiently high concentrations, particles must participate in close encounters. Such events have been observed[4] in connection with studies of suspensions. Further, more than two particles are frequently involved, leading to aggregation or coagulation, a condition which will not be of interest here. Vand[5] has estimated the effect of binary encounters on the viscosity of a suspension, but owing to the very complicated motions during the collision, a precise analytical treatment seems out of the question. Consider, for example, a suspension contained between concentric cylinders, one of which is rotating. If two particles are slightly displaced parallel to the velocity gradient, then relative motion exists. When the solids approach one another, the interaction of the flow fields gives rise to forces which are presently unknown; thus, strong mutual influences can exist without actual contact, Vand neglected those forces and considered the relatively simpler case in which the particles touch and one particle rolls around the other until the two are torn apart by the fluid motion.

Even in simplest form, the process just mentioned is very different from the kind of collisions usually treated in the kinetic theory of neutral particles. It appears, however, that there are situations in which particle–particle interactions may take place in a manner more closely resembling molecular encounters. The possibility for this behavior has been suggested most recently by researches carried out in connection with the development of solid-propellant rocket motors. One can regard the flow in the exhaust nozzle as that of a compressible fluid containing small solid particles. In first approximation, it is realistic to neglect interaction between particles and to consider them collectively as a continuum, the behavior of which is coupled to the behavior of the gas by the drag forces and heat transfer. Rannie,[6] for example, has adopted this viewpoint as the basis for analyzing the performance of a nozzle. That approach is surely valid providing the size distribution of particles is approximately uniform or the acceleration of the gas is sufficiently low that the relative velocity between particles and gas is nowhere large.

Suppose, on the contrary, that there are particles of differing sizes in an accelerating flow; the smaller particles will tend to follow the motions of the gas more closely than will the larger particles. Hence, particles of different sizes will be in relative motion,

[1] A. Einstein, *Ann. Physik* **19**, 289 (1906); reprinted in *The Theory of Brownian Movement* (Dover Publications, Inc., New York, 1956).
[2] J. M. Burgers, in *Second Report on Viscosity and Plasticity* (North-Holland Publishing Company, Amsterdam, 1938).
[3] H. L. Frisch and R. Simha, in *Rheology*, edited by F. R. Eirich (Academic Press Inc., New York, 1956), Vol. 1.
[4] R. St. J. Manley and S. G. Mason, J. Colloid Sci. **7**, 354 (1956).
[5] V. Vand, J. Phys. Coll. Chem. **52**, 277 (1948).
[6] W. D. Rannie, in *Progress in Astronautics and Rocketry*, edited by S. S. Penner and F. A. Williams (Academic Press Inc., New York, 1962), Vol. 6.

and the possibility for direct encounters between particles exists. Moreover, the greater the difference in size between particles, the more closely does the collision resemble that between two molecules although, of course, there are some important distinctions. The point is that the interactions between particles will be very unlike those occurring in the slow motion of a suspension providing the fluid is in nonuniform motion and the particles are not identical.

As a simple example, consider a situation in which the fluid is accelerating uniformly in the direction of the x axis, with speed $u = \alpha x$, and suppose that the fluid acts on the particles according to Stokes' law. Then the equation for the speed v of a spherical particle of radius σ_1 is $v\, dv/dx = 9\eta(u - v)/2\sigma_1^2\rho_s$; the viscosity of the fluid is η, and ρ_s is the density of solid material. The equation is satisfied if $v = \beta x$ with $\beta = (c/2)\{(1 + 4\alpha/c)^{\frac{1}{2}} - 1\}$ and $c = 9\eta/2\sigma_1^2\rho_s$. For small α/c, the particle speed tends to the gas speed with the square of the radius. Consequently, one can conceive of a cloud of small particles being dragged past large particles when the fluid speed is increasing.

The question of the extent to which such collisions may affect a real flow has not yet been fully investigated. Marble[7,8] has noted that within a continuum formulation there are circumstances under which particle–particle interactions may occur in the manner described, and thus may influence the transfer of momentum and energy between particles and gas. Problems involving the mutually dependent behavior of particles and gas can often be treated in that way more advantageously than by the kind of analysis presented here. However, certain aspects of two-phase flow are perhaps clarified by adopting a different description; it seems also that the kind of continuum representation discussed by Rannie and Marble may be much less appropriate when the concentration of solids becomes very large.

2. BOLTZMANN'S EQUATION FOR TWO-PHASE FLOW

The equation discussed here is exactly a form of Boltzmann's equation sometimes used to describe, among other problems, the motions of electrons and their interactions with ions and neutrals. It will be supposed that in a viscous fluid there are two classes of solid particles differing only in size. Since encounters between particles occur because of relative motion, account will be taken of collisions involving one particle of each class. This is, of course, an assumption and not a necessary consequence of the relative motion. If the distribution functions for the small and large particles are represented by $f(\mathbf{r}, \mathbf{v}, t)$, $F(\mathbf{r}, \mathbf{V}, t)$, respectively, the equation for f is

$$\frac{\partial f}{\partial t} + \nabla\cdot(\mathbf{v}f) + \nabla_v\cdot\left(\frac{\mathbf{F}}{m}f\right)$$
$$= \iint f(\mathbf{v}')F(\mathbf{V}')\left|\frac{\mathbf{v}' - \mathbf{V}'}{\mathbf{v} - \mathbf{V}'}\right|^3 |\mathbf{g}'|\,\sigma'\,d^2\Omega\,d^3\mathbf{V}'$$
$$- \iint f(\mathbf{v})F(\mathbf{V})\,|\mathbf{g}'|\,\sigma\,d^2\Omega\,d^3\mathbf{V} \tag{1}$$

The relative velocity between particles is $\mathbf{g} = -\mathbf{v} - \mathbf{V}$ and σ is the differential cross section.

If this formulation is to be useful, it must be possible to distinguish between "long-range" forces $\mathbf{F}$ and the forces acting during a collision. It will be assumed that the viscous force acting between collisions may be represented as

$$\mathbf{F} = mc(\mathbf{u} - \mathbf{v}). \tag{2}$$

For spherical particles in slow motion, $c = 9\eta/2\rho_s\sigma^2$. Use of this expression for $\mathbf{F}$ restricts in other respects the kinds of flows which can be treated. It is known, for instance, that under certain conditions a lift force acts on a particle; a necessary but not sufficient condition is that there be vorticity in the fluid far from the particle. Hence with (2) one cannot expect to discuss realistically flows in which large shear exists.

There is no need here to enumerate the various approximations involved in construction of the Boltzmann equation, but it should be noted that both upper and lower bounds on the sizes of particles are implicit. Since no account has been taken of Brownian motion, the radius of the smaller particles must be greater than, say, a few microns. On the other hand, quite distinct from whatever limit may be associated with the form assumed for the drag force, an upper limit to the permissible particle size is related to the treatment of collisions: if N is the number density of large particles having radius σ_2, then $N\sigma_2^3$ must now be large.

3. A STATIONARY DISTRIBUTION

Consider a volume of fluid extending indefinitely in all directions and in which there is a random distribution of motionless spheres, arranged with number density N. Suppose that under the action of gravity smaller spheres are falling through the

[7] F. E. Marble, in *Proceedings of the Fifth AGARD Combustion and Propulsion Colloquium, Braunschweig* (Pergamon Press, Inc., New York, 1963).

[8] F. E. Marble, Phys. Fluids **7**, 1270 (1964).

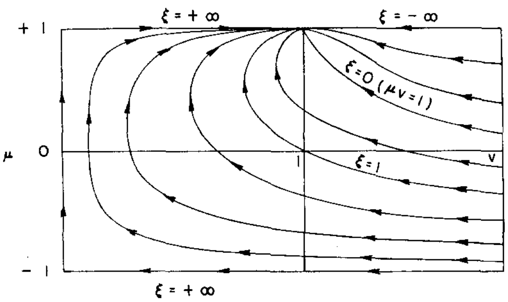

FIG. 1. Characteristic base curves (particle trajectories) in the μ–v plane.

fluid. Collisions between particles occur, but if N is large enough and the collision cross section and number density n of small spheres not too large, then one may neglect collisions between small particles. Moreover, since the transfer of kinetic energy from a small to a large particle during a collision may be reduced to an arbitrarily small value by choosing a sufficiently large difference in size, one may assume in first approximation that the large particles remain stationary. Under these conditions, it is clear that the function f is independent of $\mathbf{r}$ and t far from the source of particles and satisfies the form of Eq. (1) obtained by inserting a delta function for F, $F(\mathbf{V}) = N\delta(\mathbf{V})$. All velocities are measured relative to the large particles.

It is necessary that $\mathbf{u}$ be a constant vector, say $a\hat{k}$, where $\hat{k}$ is the unit vector in the z direction. If no fluid were entrained by the particles and the fluid is at rest, then a would be the terminal speed: $a = 2(\rho_s - \rho)g\sigma_1^2/9\eta$ for a sphere in Stokes' flow. Otherwise, a must contain the influence of the particle motion on the fluid. In any case, the equation governing f is

$$c\nabla_v \cdot \{(a\hat{k} - \mathbf{v})f\} = Nv\left[\int f(\mathbf{v}')\sigma'\, d^2\Omega - f(\mathbf{v}) \int \sigma\, d^2\Omega\right] \qquad (3)$$

providing collisions are elastic; a correction when $|\mathbf{v}'| \neq |\mathbf{v}|$ is discussed later.

A similar equation describes the distribution function for small particles being dragged by a fluid moving with constant velocity $a\hat{k}$ through an array of large particles. For example, if a gas containing the two classes of particles is moving initially at constant speed, all particles are in mechanical equilibrium with the gas; but if the gas is subject to a sudden acceleration, or deceleration (as in passage through a shock wave) to a new constant speed, the small particles move through the large particles. If the latter retain their initial velocity, then the distribution function tends to the form determined by Eq. (3).

It will be assumed that the differential cross section and hence the distribution function are independent of the azimuthal angle ϕ. In spherical coordinates (v, θ, ϕ), with $\mu = \cos\theta$ measured with respect to the gravitational field, Eq. (3) can eventually be written

$$(\mu - v)\frac{\partial f}{\partial v} + \frac{1-\mu^2}{v}\frac{\partial f}{\partial \mu} - 3f = \alpha v\left[\int_{-1}^{+1} f(v, \mu')\bar{\sigma}\, d\mu' - \beta f(v, \mu)\right], \qquad (4)$$

where

$$\bar{\sigma} = \sigma/\sigma_{12}^2; \qquad \sigma_{12}^2 = \tfrac{1}{4}(\sigma_1 + \sigma_2)^2,$$

$$\alpha = 2\pi N a \sigma_{12}^2 c^{-1}, \qquad \beta = 2\pi \int_{-1}^{+1} \sigma\, d\mu,$$

and v now is the dimensionless speed measured with respect to a. In Stokes' flow the quantity α is $8\pi\rho_s(\rho_s - \rho)N\sigma_{12}^2\sigma_1^4/81\eta^2$. As a consequence of neglecting collisions between small particles, their number density n enters the problem only as a scaling factor.

The characteristic base curves for the left-hand side of (4) form the one-parameter family

$$(1 - \mu v)/v(1 - \mu^2)^{\frac{1}{2}} = \xi. \qquad (5)$$

These curves, sketched in Fig. 1, comprise the possible trajectories of a small particle given any initial conditions and in the absence of collisions. The solution $f(v, \mu)$ may be represented as a surface above the v–μ plane; since the speed of a particle cannot exceed the terminal speed, only the region $v \leq 1$ is of interest.

Equation (3) can be easily integrated only if the differential cross section behaves in special ways: since the collisional processes are unknown, the simplest assumption will be made. Namely, if σ is constant, as for collisions between rigid spheres, or if σ depends only on the direction of a small particle after collision, one finds

$$f = \psi\left(\frac{\xi + s}{\xi + s_1}\right)^3 e^{-\alpha I(s)} \cdot \left\{1 + \alpha K \int_{s_1}^{s} \left(\frac{\xi + s}{\xi + s_1}\right)^3 \frac{(1 + y^2)^{\frac{1}{2}}}{(\xi + y)^2} e^{\alpha I(y)}\, dy\right\}, \qquad (6)$$

with

$$I(s) = \int_{s_1}^{s} \frac{(1 + x^2)^{\frac{1}{2}}}{(\xi + x)^2}\, dx, \qquad (7)$$

$$K(\xi, \alpha) = \int_{-\infty}^{\infty} \left(\frac{\xi + s}{\xi + s_1}\right)^3 e^{-\alpha I(s)} \frac{\bar{\sigma}\, ds}{(1 + s^2)^{\frac{3}{2}}}$$
$$\cdot \left\{1 - \alpha \int_{-\infty}^{\infty} \left[\left(\frac{\xi + s}{\xi + s_1}\right)^3 e^{-\alpha I(s)} \int_{s_1}^{s} \left(\frac{\xi + s_1}{\xi + y}\right)^3 \right.\right.$$
$$\left.\left. \cdot \frac{(1 + y^2)^{\frac{1}{2}}}{(\xi + y)^2} e^{\alpha I(y)}\, dy\right] \frac{\bar{\sigma}\, ds}{(1 + s^2)^{\frac{3}{2}}}\right\}^{-1}. \qquad (8)$$

The integration has been carried out in the variable $s = \mu/(1 - \mu^2)^{\frac{1}{2}}$ after elimination of $\partial f/\partial v$ by transformation of variables from (v, μ) to (ξ, s).

Although Eq. (6) is an integral of Eq. (4), it does not represent the complete solution; for in the limit $\alpha \to 0$, Eq. (6) becomes $\psi(\xi; \alpha \to 0)(\xi + s)^3/(\xi + s_1)$. However, $\alpha \to 0$ is the collisionless limit, and in that case, the distribution function is simply a delta function placed at $v = \mu = 1$. The function $\psi(\xi + s)^3/(\xi + s_1)^3$ is an unspecified combination of the characteristic base curves and in general cannot be put in the form of a delta function. On the other hand, one can easily show, by setting $f = [\delta(v - 1)/v^2]\, \delta(\mu - 1)$ in Eq. (4) and integrating over velocity space, that both sides of the equation are identically satisfied. Thus, the special solution

$$\Delta = [n_0(\alpha)/2\pi][\delta(v - 1)/v^2]\delta(\mu - 1), \qquad (9)$$

which is acceptable on physical grounds, is also an integral of the equation for f in the sense just described and may be added to (6) to give the complete solution. The number density of particles falling vertically is n_0, which depends on α in such a way that for $\alpha = 0$, $n_0 = n$. Let $\mathfrak{F}$ denote the right-hand side of (6), and the complete solution for the distribution of small particles is $f = \mathfrak{F} + \Delta$. This combination resembles the kind of result one finds in free-molecule kinetic theory. The second part is the distribution in "free molecule" flow, and contains all particles falling vertically downward with

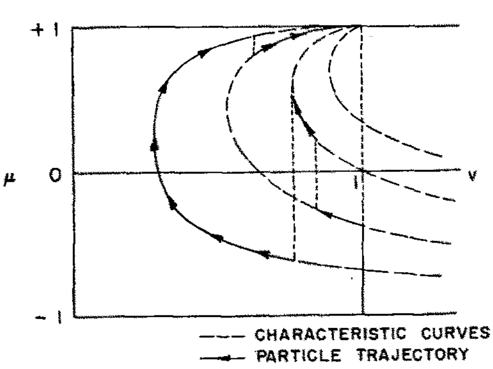

FIG. 2. A particle trajectory in the μ–v plane, showing four elastic collisions.

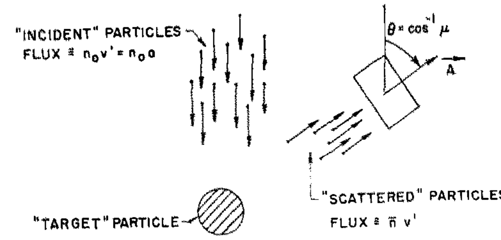

FIG. 3. Scattering of small particles by a single target particle.

the terminal speed a; $\mathfrak{F}$ contains all particles which have suffered collisions.

The motion of a representative point P is sketched in Fig. 2. In (f, v, μ) space, P moves in the surface $\mathfrak{F}(s, \xi, \alpha)$, but the projection of its path may have the form shown. Between collisions, P follows a characteristic curve; the irregular jumping of all points P between curves $\xi =$ const is represented by $\mathfrak{F}$. In this steady-state situation, there is a flow of particles from Δ to $\mathfrak{F}$ due to collisions and Δ is replenished owing to the action of long-range forces acting between collisions.

The function ψ is found by specifying the values of f along some line (not a characteristic curve) in the v–μ plane. Evidently, one should consider the values on $v = 1$, where, excluding the point $\mu = v = 1$, f is proportional to the differential cross section. Any particle moving with the terminal speed is either falling vertically, and hence contained in Δ, or since all collisions are assumed elastic, has just been transferred from Δ to $\mathfrak{F}$ after colliding with a large particle. A simple calculation of the scattering of particles in Δ by a single target particle, as sketched in Fig. 3, gives the balance $\bar{n}v'A = n_0 v'\sigma$. The area A available to scattered particles is the average area associated with a single target and is $N^{\frac{2}{3}}$. Thus the distribution function for the scattered particles is $\bar{n}v'N^{\frac{2}{3}}\, \delta(v' - a)/2\pi v'^2$; accordingly, the function ψ is proportional to $\bar{n} = n_0 N^{\frac{2}{3}}\sigma$:

$$\psi = (\epsilon/2\pi)n_0 N^{\frac{2}{3}}\sigma(\mu). \qquad (10)$$

The quantity ϵ has magnitude unity, but the dimensions of a volume element in velocity space. This expression can also be obtained from the "gain" part of the collision integral.

4. SOME APPROXIMATIONS TO THE STATIONARY DISTRIBUTION

Because of the similarity between Eq. (10) and certain equations encountered in the descriptions of other problems, approximate methods of solution are suggested. Thus, for example, if one assumes that the particles collide roughly in the manner of rigid spheres, the Legendre polynomials $P_n(\mu)$ are eigenfunctions for the integral on the right-hand side of Eq. (3). One is tempted to expand the continuous part of f as a series of terms having the form $V_n(v)P_n(\mu)$; as in similar situations, the coefficients $V_n(v)$ can be evaluated only after arbitrarily truncating the series. In view of the form of the exact solution, the appearance of this difficulty is hardly surprising, and for this problem, such a representation is certainly not suitable.

If no special form is assumed for $\bar{\sigma}$, then the most obvious approach is based on an expansion in powers of α. Set

$$f = \sum_{n=0}^{\infty} \alpha^d f^n + \Delta \tag{11}$$

and write Eq. (3) for f as

$$\mathfrak{D}f = \alpha\{\mathcal{G}(f) - \mathcal{L}(f)\}. \tag{12}$$

The nth term of the expansion is found as the solution to

$$\mathfrak{D}(f)^n = \mathcal{G}(f^{n-1}) - \mathcal{L}(f^{n-1}). \tag{13}$$

Moreover, it is also a straightforward matter to expand the complete solution in powers of α; the result, of course, coincides with the expression (11), and one may write these, including terms of order α^2, as

$$f \cong \psi[(\xi + s)/(\xi + s_1)]^3\{1 + \overset{(1)}{\alpha}[\overset{(2)}{K_0M} - \overset{(1)}{I}]$$
$$+ \alpha^2[\overset{(3)+(2)}{MK_1} + \overset{(2)}{K_0}(\overset{(2)}{N} - \overset{(1)}{MI}) + \overset{(0)}{\tfrac{1}{2}I^2}]\} + \Delta \tag{14}$$

(numbers above the terms in this equation will be explained shortly), with the abbreviations

$$M = \int_{s_1}^{s} \left(\frac{\xi + s_1}{\xi + y}\right)^3 \frac{(1 + y^2)^{\frac{1}{2}}}{(\xi + y)^2}\, dy,$$

$$N = \int_{s_1}^{s} I\, \frac{(1 + y^2)^{\frac{1}{2}}}{(\xi + y)^2} \left(\frac{\xi + s_1}{\xi + y}\right)^3 dy,$$

$$K_0 = \int_{-\infty}^{\infty} \left(\frac{s + \xi}{s_1 + \xi}\right)^3 \frac{\bar{\sigma}\, ds}{(1 + s^2)^{\frac{3}{2}}},$$

$$K_1 = K_0 \int_{-\infty}^{\infty} M\left(\frac{\xi + s}{\xi + s_1}\right)^3 \frac{\bar{\sigma}\, ds}{(1 + s^2)^{\frac{3}{2}}}$$
$$\int_{-\infty}^{\infty} I\left(\frac{\xi + s}{\xi \quad s_1}\right)^3 \frac{\bar{\sigma}\, ds}{(1 + s^2)^{\frac{3}{2}}}$$
$$= K_0k_1 - k_2.$$

The expansion is valid only if αI is small. But for the case treated here, $s_1 = (1 - \xi^2)/2\xi$; and $\exp(-\alpha I)$ has a term $\exp[-\alpha(1 + s^2)/(\xi + s)] \equiv \exp[-\alpha v/(1 - \mu^2)^{\frac{1}{2}}]$. Consequently, since $I \to \infty$ for $\mu \to \pm 1$, it is not possible to obtain a simple expansion, in powers of α, convergent over the entire range of interest.

Some difficulty is encountered if one attempts to interpret the terms f^n in the expansion (14). The governing equation (12) represents the balance of representative points for an element of volume $dV_p = d^3\mathbf{x}\, d^3\mathbf{v}$ of phase space: the rate of change of representative points, or "particles in f" evaluated at dV_p, is equal to the difference between the rate at which particles in f are scattered into dV_p, and the rate at which particles in f are scattered out of dV_p. But a corresponding interpretation applied to the left-hand side of Eq. (13) does not make sense: the rate of change of the number of particles in f^n evaluated at dV_p is equal to the difference between the rate at which particles are scattered into f^n from f^{n-1} and the rate at which particles in dV_p are scattered out of f^{n-1}. The meaning of the equation, and hence f^n, is destroyed by the form of the loss term.

Although the approximate solution by expansion in powers of α is correct, it is, of course, not unique; and in particular, it is possible to find a series expansion for which the individual terms and the governing equations have very simple interpretations. The basis for this expansion is the observation, illustrated by Fig. 2, that in the course of its motion, a particle may be indexed according to the number of collisions it has suffered since falling vertically downward. Let f_n denote that part of the complete distribution, f, containing all particles which have had n encounters since they were scattered out of Δ; thus, the expansion of f is

$$f = \sum_{n=1}^{\infty} f_n + \Delta. \tag{15}$$

Under the action of collisions, there is a flow of particles from Δ to f_1 to f_2 . . . , and finally back into Δ. Clearly, the gain of particles in f_n is associated with scattering out of f_{n-1} so that one is able immediately to write down the sequence of equations for the f_n:

$$\mathfrak{D}(f_n) = \alpha\{\mathcal{G}(f_{n-1}) - \mathcal{L}(f_n)\}. \tag{16}$$

If the series is truncated, then the equation for the last f_n must contain no loss term, for the passage of particles into Δ is associated with the left-hand side. Since particles for which $v = 1$, $\mu \neq 1$ must have suffered only one collision, the initial conditions on the f_n are $f_1 = \psi$ and $f_n = 0$ for $\mu \geq 0$. One can easily determine that f_n contains only terms of order α^{n-1} and higher; to terms of order α^2, the set (16) yields $f_0 = \Delta$ and

$$f_1 = \psi\left(\frac{\xi + s}{\xi + s_1}\right)^3 e^{-\alpha I} \cong \psi\left(\frac{\xi + s}{\xi + s_1}\right)^3\left[1 - \alpha I + \alpha^2 \frac{I^2}{2}\right],$$

$$f_2 \cong \psi[(\xi + s)/(\xi + s_1)]^3 \cdot[\alpha K_0 M + \alpha^2(K_0 N - K_0 MI - Mk_2],$$

$$f_3 \cong \psi[(\xi + s)/(\xi + s_1)]^3[\alpha^2 K_0 M k_1].$$

Substitution of these results into Eq. (15) and rearrangement according to powers of α leads once again to Eq. (14): the numbers in parentheses above that equation are the indices associated with the f_n from which the terms came.

In contrast to the expansion in powers of α, the leading term (f_1) of the representation (15) is valid everywhere. Moreover, one is able now to attach a meaning to the successive terms of the expansion in powers of α, and it is apparent that, so far as the history of the individual particles is concerned, the expansion (11) is not the most fortunate representation. For f^0 contains only, but not all, particles which have suffered one collision; f^1 contains only, but not all, particles which have suffered one or two collisions; and so on. Particles which have had m encounters since belonging to Δ (i.e., those in f_m) are distributed among all f^n for $n \geq m - 1$. Inclusion of higher-order terms in α thus corresponds to accounting for "more collisions," but not in a simple fashion.

There is associated with each of the above expansions an interation procedure. Let f be computed as the limit of the sequence $f^{(n)}$,

$$f = \lim_{N\to\infty} \{f^{(0)}, f^{(1)}, \ldots, f^{(N)}\}, \tag{17}$$

where $f^{(N)}$ is the solution to

$$\mathfrak{D}f^{(N)} = \alpha\{\mathcal{G}f^{(N-1)} - \mathcal{L}f^{(N-1)}\} \tag{18}$$

and

$$\mathfrak{D}f^{(0)} = 0. \tag{19}$$

It is easy to verify that $f^{(N)}$ is identical with the truncated series $\sum_{n=0}^{N} f^n$ of the expansion (11). In studies of rarefied gas flows, this procedure is commonly referred to as Knudsen iteration. The collisionless part of f (i.e., Δ) must be added to this expansion, and the first iterate, $f^{(0)}$, is not the collisionless distribution but is the continuous solution to the homogeneous equation.

A different iteration procedure is based on the sequence $f_{(n)}$,

$$f = \lim_{N\to\infty} \{f_{(0)}, f_{(1)}, \ldots, f_{(N)}\}, \tag{20}$$

where now $f_{(N)}$ is the solution to

$$\mathfrak{D}f_{(N)} = \alpha\{\mathcal{G}(f_{(N-1)}) - \mathcal{L}(f_{(N)})\}. \tag{21}$$

Use of this procedure has been discussed, for example, by Willis[9] and Narasimha.[10] If, in the present situation, one chooses $f_{(0)}$ to be the collisionless solution, Δ, then $f_{(N)}$ is exactly the solution (15) with the series truncated after N terms. Thus, unlike the Knudsen iteration scheme, the complete solution is obtained.

5. DRAG FORCE BETWEEN LARGE AND SMALL PARTICLES

In connection with the study of the influence of collisions within a continuum representation, it is necessary to have a formula for the average force, per unit volume, exerted between the two classes of particles. By use of an argument based on elementary kinetic theory, Marble[8] has introduced an expression tied essentially to the assumption that collisions occur only between particles which have the speed appropriate to local conditions existing in the absence of collisions. It is then necessary that some sort of "equilibration time" be much less the average time between collisions, and, moreover, that the corresponding "equilibration length" be much smaller than the scale of the flow field. The approximations have been discussed by Marble and may be applied here. The complete Boltzmann equation (1), written for steady flow with normalized variables, is

$$(\lambda_v/L)\nabla\cdot(\mathbf{v}f) + \nabla_v\cdot\{\mathbf{u} - \mathbf{v})f\} = \alpha\{\mathcal{G}(f) - \mathcal{L}(f)\}, \tag{22}$$

in which L is the macroscopic scale and $\lambda_v = a/c$ is the equilibration length; the speed a is now a typical "equilibrium" speed. Clearly, if λ_v/L is small, then one may try an expansion in powers of λ_v/L; the first term of the expansion is governed by the same equation as that treated in the two preceding sections, except that the physical coordinates appear parametrically in $\mathbf{u}$. The distribution function corresponding to the assumption used by Marble is clearly the sum of Δ and f_1 in Eq. (15) (the velocity $a\hat{k}$ must, of course, be properly interpreted), since $\Delta + f_1$ contains all particles which are moving in "local equilibrium" and all

[9] D. R. Willis, Princeton University Aero. Eng. Report No. 442 (1958).

[10] R. Narasimha, Ph.D. thesis, California Institute of Technology (1961).

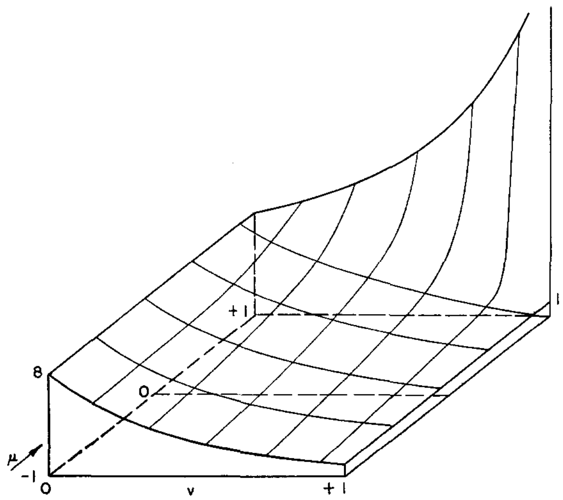

FIG. 4. Showing the distribution function f_1/ψ when single collisions only are included.

particles which have suffered one collision only. In the limit $\alpha \to 0$, the exact integral for this case yields

$$f \cong \psi[(\xi + s)/(\xi + s_1)]^3 + \Delta. \tag{23}$$

Evaluation of the drag force proceeds exactly as in the kinetic theory of gases and gives

$$F_{12} = n_0 N m^* \sigma_m \, |g'|^2 \tag{24}$$

in the direction of the "equilibrium" velocity of the small particles; σ_m is the cross section for momentum transfer, and m^* is the reduced mass.

That there are some small particles not in Δ influences the value of n_0. If n is specified and f is approximated by Eq. (23), then

$$n_0 = n\left[1 + 2\pi \int_0^1 \int_{-1}^{+1} \frac{\psi}{N_0} \left(\frac{\xi + s}{\xi + s_1}\right)^3 d\mu \, dv\right].$$

For the simplest case of elastic collisions, $s_1(\xi)$ is determined by setting the particle speed equal to the local equilibrium speed in Eq. (15), $v = 1$; then $s_1(\xi) = (1 - \xi^2)/2\xi$. The integral is most easily carried out in the variables (ξ, s) and one finds

$$n_0 = n/(1 + \tfrac{2}{3}\pi^2 N^{\frac{1}{3}}\sigma). \tag{25}$$

A sketch of the function $(\xi + s)^3/(\xi + s_1)^3$ is shown in Fig. 4.

6. AN APPROXIMATE ACCOUNTING FOR THE INFLUENCE OF INELASTIC COLLISIONS

The assumption that the speed of a small particle is not altered by a collision depends on two circumstances; the particles exchange no kinetic energy during a collision, and there is no dissipation of energy. A possible situation in which the energy dissipated, by viscous forces acting on a small particle, is precisely compensated by a transfer of kinetic energy from the large particle, may also arise. Since no attempt is being made to compute the distribution function for the large particles, one is able to estimate simultaneously the effects of both the transfer of kinetic energy ("recoil") and the dissipation of energy during a collision. (In fact, one cannot do otherwise if only the velocities of a small particle before and after a collision are specified.) It follows that the ratio v'/v can be treated as a parameter measuring the energy loss in a collision.

When $\mathbf{v}' = 0$, the only change in Eq. (3) is that a factor $|v'/v|^4$ appears in the first integral. All considerations of the angular distribution appear in $\bar{\sigma}$. For the simplest case, when $(v'/v)^2 = 1 + \epsilon$, with ϵ a constant, the only change in the previous results is that s_1 is now determined by the condition $(1 + s_1^2)^{\frac{1}{2}}/(\xi + s_1) = (1 + \epsilon)^{-\frac{1}{2}}$. To first order in ϵ,

$$s_1 = (1 - \xi^2)/2\xi + \epsilon(1 + \xi^2)^2/8\xi^2$$

and for $\alpha \to 0$,

$$f_1 \cong \psi \frac{8\xi^3(\xi + s)^3}{(1 + \xi^2)^3} \frac{1}{[1 + \epsilon(1 + \xi^2)/4\xi^2]}, \tag{26}$$

showing that the distribution function is not only confined to the region $0 = v = (1 + \epsilon)^{-\frac{1}{2}}$ but is also decreased from the values sketched in Fig. 4. Consequently, a greater fraction of the particles are contained in Δ. The reason for this is evidently that, upon becoming part of $\mathfrak{F}$, each particle is in a sense closer to Δ since its speed has already been reduced by the factor $(1 + \epsilon)^{-\frac{1}{2}}$.

ACKNOWLEDGMENT

This work was supported by National Science Foundation Grant GP-713.